limits

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} f(x)$ exist. Then

Sum Law	$\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$
Difference Law	$\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$
Constant Multiple Law	$\lim_{x\to a}\left(cf(x)\right)=c\lim_{x\to a} f(x)$
Product Law	$\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$
Quotient Law	$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x)\neq 0$
Power Law	$\lim_{x\to a}\left(f(x)\right)^n=\left(\lim_{x\to a} f(x)\right)^n$
Root Law	$\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$, where n is a positive integer
Constant Law	$\lim_{x\to a} c=c$
Direct Substitution Law	$\lim_{x\to a} f(x)=f(a)$

L'Hospital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ a is a number, ∞ or $-\infty$

Find $\lim_{x\to -2}\frac{x+2}{x^2+3x+2}$

(step 1) **Plug in to evaluate the limit, if possible**

$$\lim_{x\to -2}\frac{x+2}{x^2+3x+2}=\frac{(-2)+2}{(-2)^2+3(-2)+2}=\frac{0}{0}\ \text{(indeterminate)}$$

(step 2) **Apply L'Hospital's rule and reevaluate the limit**

$$\lim_{x\to -2}\frac{\frac{d}{dx}[x+2]}{\frac{d}{dx}[x^2+3x+2]}=\lim_{x\to -2}\frac{1}{2x+3}$$

$$\lim_{x\to -2}\frac{1}{2x+3}=\frac{1}{2(-2)+3}=\frac{1}{-1}=\boxed{-1}$$

Calculating $\lim_{x\to a} f(x)$

A. Direct substitution
Try to evaluate the function directly.

$f(a)$

- $f(a)=\frac{b}{0}$ where ***b*** is not zero — **B. Asymptote (probably)** example: $\lim_{x\to 1}\frac{1}{x-1}$. Inspect with a graph or table to learn more about the function at ***x=a***.
- $f(a)=b$ where ***b*** is a real number — **C. Limit found (probably)** example: $\lim_{x\to 3}x^2=(3)^2=9$
- $f(a)=\frac{0}{0}$ — **D. Indeterminate form** example: $\lim_{x\to -1}\frac{x^2-x-2}{x^2-2x-3}$

Try rewriting the limit in an equivalent form.

- **E. Factoring** example: $\lim_{x\to -1}\frac{x^2-x-2}{x^2-2x-3}$ can be reduced to $\lim_{x\to -1}\frac{x-2}{x-3}$ by factoring and cancelling.
- **F. Conjugates** example: $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$ can be rewritten as $\lim_{x\to 4}\frac{1}{\sqrt{x}+2}$ using conjugates and cancelling.
- **G. Trig identities** example: $\lim_{x\to 0}\frac{\sin(x)}{\sin(2x)}$ can be rewritten as $\lim_{x\to 0}\frac{1}{2\cos(x)}$ using a trig identity.

Try evaluating the limit in its new form.

H. Approximation
When all else fails, graphs and tables can help approximate limits.

rules:

Derivative	Integral (Antiderivative)
$\frac{d}{dx}n=0$	$\int 0\,dx=C$
$\frac{d}{dx}x=1$	$\int 1\,dx=x+C$
$\frac{d}{dx}x^n=nx^{n-1}$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$
$\frac{d}{dx}e^x=e^x$	$\int e^x\,dx=e^x+C$
$\frac{d}{dx}\ln x=\frac{1}{x}$	$\int\frac{1}{x}\,dx=\ln x+C$
$\frac{d}{dx}n^x=n^x\ln x$	$\int n^x\,dx=\frac{n^x}{\ln n}+C$
$\frac{d}{dx}\sin x=\cos x$	$\int\cos x\,dx=\sin x+C$
$\frac{d}{dx}\cos x=-\sin x$	$\int\sin x\,dx=-\cos x+C$
$\frac{d}{dx}\tan x=\sec^2 x$	$\int\sec^2 x\,dx=\tan x+C$
$\frac{d}{dx}\cot x=-\csc^2 x$	$\int\csc^2 x\,dx=-\cot x+C$
$\frac{d}{dx}\sec x=\sec x\tan x$	$\int\tan x\sec x\,dx=\sec x+C$
$\frac{d}{dx}\csc x=-\csc x\cot x$	$\int\cot x\csc x\,dx=-\csc x+C$
$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}$	$\int\frac{1}{\sqrt{1-x^2}}\,dx=\arcsin x+C$
$\frac{d}{dx}\arccos x=-\frac{1}{\sqrt{1-x^2}}$	$\int-\frac{1}{\sqrt{1-x^2}}\,dx=\arccos x+C$
$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$	$\int\frac{1}{1+x^2}\,dx=\arctan x+C$
$\frac{d}{dx}\text{arc}\cot x=-\frac{1}{1+x^2}$	$\int-\frac{1}{1+x^2}\,dx=\text{arc}\cot x+C$
$\frac{d}{dx}\text{arc}\sec x=\frac{1}{x\sqrt{x^2-1}}$	$\int\frac{1}{x\sqrt{x^2-1}}\,dx=\text{arc}\sec x+C$
$\frac{d}{dx}\text{arc}\csc x=-\frac{1}{x\sqrt{x^2-1}}$	$\int-\frac{1}{x\sqrt{x^2-1}}\,dx=\text{arc}\csc x+C$

Differentiation Rules	
Constant Rule	$\frac{d}{dx}[c]=0$
Power Rule	$\frac{d}{dx}x^n=nx^{n-1}$
Product Rule	$\frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))]=f'(g(x))g'(x)$

critical point: f'(c)=0 or DNE

incr/decr:
1) find crit pts of f
2) make sign chart for f'
3) plug in values to determine if incr or decr

find abs max/min of f on [a,b]
1) evaluate f @ x = a,b (endpoints)
2) find crit pts on [a,b] (set der=0) & evaluate f @ crit pts
3) compare values (largest=abs max; smallest=abs min

concavity:
1) f''(x)
2) sign chart for f''
3) test for concavity
4) inflection pt = where concavity switches

Maxim/Minim:
1) constraint
2) set der = 0
3) plug value into 2nd der
+?. max value
−?. min value
4) plug value into constraint to find dimensions

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right)=n\left[f(x)\right]^{n-1}f'(x)$
2. $\frac{d}{dx}\left(e^{f(x)}\right)=f'(x)e^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right)=\frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right)=f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right)=-f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right)=f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right)=f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right)=\frac{f'(x)}{1+\left[f(x)\right]^2}$

U-Subst.

$\int x^2(x^3-7)^3dx$	
$\int x^2u^3dx$	Substitute u
$\int x^2u^3\frac{1}{3x^2}du$	Substitute du
$\int\frac{1}{3}u^3du$	Cancel the x^2
$\frac{1}{3}\int u^3du$	Factor out the 1/3

integral shortcuts:

$$\int[f(x)]^n\cdot f'(x)dx=\frac{[f(x)]^{n+1}}{n+1}+C$$

$$\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c$$

definite integral:

$$\int_a^b f(x)dx=F(b)-F(a)$$

velocity:
<0: right → left
>0: left → right

***u* Substitution :** The substitution $u=g(x)$ will convert $\int_a^b f(g(x))g'(x)dx=\int_{g(a)}^{g(b)}f(u)\,du$ using $du=g'(x)dx$. For indefinite integrals drop the limits of integration.

Ex. $\int_1^2 5x^2\cos(x^3)dx$

$u=x^3\ \Rightarrow\ du=3x^2dx\ \Rightarrow\ x^2dx=\frac{1}{3}du$

$x=1\ \Rightarrow\ u=1^3=1\ ::\ x=2\ \Rightarrow\ u=2^3=8$

$\int_1^2 5x^2\cos(x^3)dx=\int_1^8\frac{5}{3}\cos(u)du$

$=\frac{5}{3}\sin(u)\Big|_1^8=\frac{5}{3}(\sin(8)-\sin(1))$

Find the linearization of the function $f(x)=\sqrt[3]{x}$ at $a=-8$ and use it to approximate the number -8.1

1. Plug in $a=-8$ for x and solve for y to find ordered pair	$f(a)=\sqrt[3]{x}$ $f(-8)=\sqrt[3]{-8}=-2$ $(-8,-2)$
2. Take the derivative to find the slope of the tangent line	$f(x)=\sqrt[3]{x}=x^{1/3}$ $\frac{dy}{dx}=\frac{1}{3}x^{-2/3}=\frac{1}{3x^{2/3}}$
3. Plug in the ordered pair from step 1 and solve for the slope: $\frac{dy}{dx}\Big\|_{x=-8,y=-2}$	$\frac{dy}{dx}=\frac{1}{3(-8)^{2/3}}$ $m=\frac{1}{3(-2^2)^{2/3}}=\frac{1}{3(-2)^2}$ $m=\frac{1}{12}$
4. The Linearization is found by substituting the ordered pair and slope found in step 3 into Point-Slope Form	$y-y_1=m(x\quad x_1)$ $y-(-2)=\frac{1}{12}(x-(-8))$ $y+2=\frac{1}{12}(x+8)$ $y=\frac{1}{12}x-\frac{4}{3}$
5. Find the Linear Approximation of the number -8.1, plug it into the equation of the tangent line	$y=\frac{1}{12}x-\frac{4}{3}$ $y=\frac{1}{12}(-8.1)-\frac{4}{3}$ $y=-2.008$

Riemann's sum:

REVIEW

The Limit Equation for Riemann's Sum

$$\Delta x = \frac{b-a}{n}$$

$$x_K = a + (\Delta x)k$$

Upper limit of summation: It tells us to end with $k = n$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Index of summation

Lower limit of summation: It tells us to start with $k = 1$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\Delta x)[f(x_k)]$$

It is more traditional to use $k = 0$ for *left* or *midpoint* sums, and $k = 1$ for *right* sums.

11/22/2018 9:52 PM   §4.2B: Limits of Riemann's Sum   4

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x,$$

↰ left

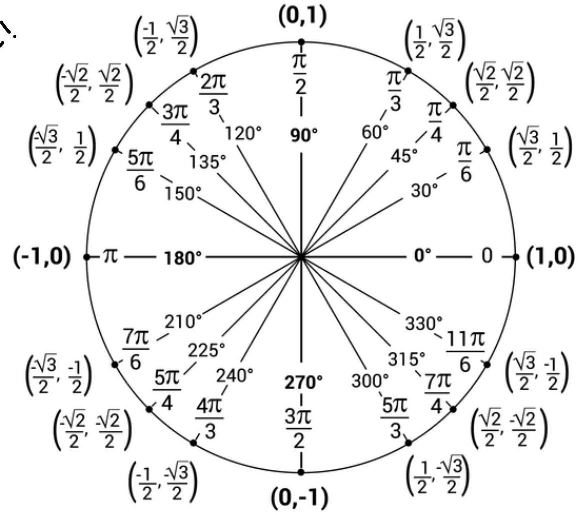
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x,$$

↰ right

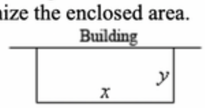
$$M_n = f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + \cdots + f(\overline{x}_n)\Delta x = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x,$$

↰ midpoint

unit circle!



Ex. We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize $A = xy$ subject to constraint of $x + 2y = 500$. Solve constraint for x and plug into area.

$$x = 500 - 2y \quad \Rightarrow \quad A = y(500-2y) = 500y - 2y^2$$

Differentiate and find critical point(s).

$$A' = 500 - 4y \quad \Rightarrow \quad y = 125$$

By 2nd deriv. test this is a rel. max. and so is the answer we're after. Finally, find x.

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

optimization

1) draw diagram & introduce variables
2) write down conditions on the variables
3) determine the quantity to be maximized/minimized
4) write down an equation for the quantity to max/min & reduce # of variables
5) use der to find max/min value

① point
② slope
③ point-slope

find the equation of the tangent line to the graph of $y = \cot x$ @ $x = \pi/4$

point: $(\pi/4, \cot(\pi/4)) = (\pi/4, 1)$

slope: $y'(\pi/4)$: $y' = (\cot x)' = -\csc^2 x$

$-\csc^2 x = \frac{-1}{\sin^2(\pi/4)} = -2$

$y - y_1 = m(x - x_1)$

$y - 1 = (-2)(x - \pi/4) \Rightarrow \boxed{y = -2x + \pi/2 + 1}$

AREA:

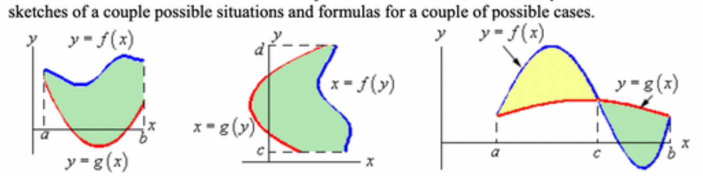
Applications of Integrals

Net Area : $\int_a^b f(x)\,dx$ represents the net area between $f(x)$ and the x-axis with area above x-axis positive and area below x-axis negative.

Area Between Curves : The general formulas for the two main cases for each are,

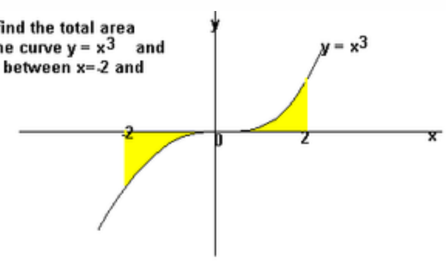
$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}]\,dx \quad \& \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}]\,dy$$

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



$$A = \int_a^b f(x) - g(x)\,dx \qquad A = \int_c^d f(y) - g(y)\,dy \qquad A = \int_a^c f(x) - g(x)\,dx + \int_c^b g(x) - f(x)\,dx$$

Example: find the total area between the curve $y = x^3$ and the x-axis between x=-2 and x=2.



If we simply integrated x^3 between -2 and 2, we would get:

$$\left[\frac{x^4}{4}\right]_{-2}^{2} = 4 - 4 = 0$$

So instead, we have to split the graph up and do two separate integrals.

$$\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 16/4 - 0 = 4$$

$$\int_{-2}^0 x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^0 = 0 - 16/4 = -4 \quad \text{(so area is 4).}$$

We then add these two up to get: <u>8 units^2</u>

RULES FOR TRANSFORMATIONS OF FUNCTIONS

If $f(x)$ is the original function, $a > 0$ and $c > 0$:

Function	Transformation of the graph of $f(x)$
$f(x)+c$	Shift $f(x)$ upward c units
$f(x)-c$	Shift $f(x)$ downward c units
$f(x+c)$	Shift $f(x)$ to the left c units
$f(x-c)$	Shift $f(x)$ to the right c units
$-f(x)$	Reflect $f(x)$ in the x-axis
$f(-x)$	Reflect $f(x)$ in the y-axis
$a \cdot f(x),\ a > 1$	Stretch $f(x)$ vertically by a factor of a.
$a \cdot f(x),\ 0 < a < 1$	Shrink $f(x)$ vertically by a factor of a.
$f(ax),\ a > 1$	Shrink $f(x)$ horizontally by a factor of $\frac{1}{a}$.
$f(ax),\ 0 < a < 1$	Stretch $f(x)$ horizontally by a factor of $\frac{1}{a}$.

linear appr.

estimate the value of $\frac{1}{2.1}$

$f(x) = \frac{1}{x}$ $a = 2$

$f(2.1) = \frac{1}{2.1} \approx L(2.1)$

$L(x) = f(a) + f'(a)(x-a)$

$f(2) + f'(2)(x-2)$

$f(2) = 1/2$ $f'(x) = -x^{-2} = \boxed{-\frac{1}{4}}$

If $f(3) = 2$ + $f'(3) = 5$, est. $f(3.2)$

$\underbrace{f(a+h)}_{f(3.2)} \approx \underbrace{f(a)}_{f(3)} + \underbrace{f'(a)}_{f'(3)} \cdot h$

$h = 3.2 - 3 = 0.2$

$f(3.2) \approx f(3) + f'(3) \cdot 0.2 = 3$

$f(3.2) \approx 3$